Thermodynamic Functions of a Parabosonic System of Second Order

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Abstract. This study aims to determine the thermodynamic properties of a paraboson system of order two. The thermodynamic properties to be searched include the grand canonical partition function (GCPF), *Z*, and the average number of particles, *N*. These parastatistics systems is a more general form compared to a quantum statistical distribution that has been known previously, i.e. Bose-Einstein statistics (BE). Starting from the recursion relation of the GCPF for paraboson system of order two, recursion linkages for some simple thermodynamic functions for paraboson system of order two may be derived. The recursion linkages are then used to calculate the thermodynamic function of the model system of identical particles with limited energy levels which is similar to the harmonic oscillator. The results from *Z* show that the thermodynamics properties of the paraboson system of order two can be derived and have a similar shape with paraboson system of order one (boson). The similarity of the graph shows similar thermodynamic properties.

1. Introduction

Given the principles of quantum mechanics, there is no necessity that the statistics of the particles must satisfy Bose-Einstein (BE) or Fermi-Dirac (FD) statistics. But both of the latter statistics have been proven through various experiments. Particles that satisfy Bose-Einstein statistics are called boson which has symmetrical wave functions under the exchange of any two particles. Examples include photons, alpha particles, and Helium atoms. On the other hand, fermions are particles that satisfy Fermi-Dirac statistics and Pauli's exclusion principle that is particles whose wave functions are anti-symmetric under the exchange of any two particles. Examples include protons, neutrons, and electrons.

Many physicists attempt to formulate a more general statistics than the above existing statistics, either by creating new types of statistics or by generalizing Bose and Fermi statistics. Some type of statistics other than Bose and Fermi statistics are already introduced, such as null statistics, ortho-fermi statistics, Hubbard statistics, and others [1]. Moreover, statistics resulted from a generalization of existing statistics include intermediate statistics, parastatistics, infinite statistics, paronic statistics, anyon statistics, and others [2].

Parastatistics which is first introduced by Green [3], is the first consistent generalization of BE and FD quantum statistics called paraboson and parafermion, respectively. Parastatistics satisfies the tri-linear commutation relation for the operator's creation and annihilation of particles and meets the principles of cluster decomposition [4]. Because of this, although there is no indication that the

fundamental particles present today satisfy the parastatistics principles, this theory remains interesting for further investigation. Many physicists have attempted to obtain the corresponding physical quantities for the parastatistics system, in particular, the grand canonical partition function (GCPF). Obtaining the GCPF will reveal the thermodynamic properties of the parastatistics system, so it can serve as a basis for clarifying whether a physical system complies with these statistics or not.

Parastatistics is the first consistent generalization of the Bose and Fermi quantum statistics. In its formulation, the standard bi-linier commutation (Bose) and anti-commutation (Fermi) relationships for the particle creation and annihilation operators are

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right]_q \equiv \hat{a}_i\hat{a}_j^\dagger - q\hat{a}_j^\dagger\hat{a}_i = \delta_{ij}, \tag{1}$$

with $q = +1$ for Bose statistics and $q = -1$ for Fermi statistics, which is substituted with the following tri-linear relation

$$\left[\hat{a}_i^\dagger\hat{a}_j \pm \hat{a}_j\hat{a}_i^\dagger, \hat{a}_k^\dagger\right] = \frac{2}{p}\delta_{jk}\hat{a}_i^\dagger, \tag{2}$$

with p is a positive integer which is the parastatistics order.

Hartle et al. (1970) have shown that parastatistics is a quantum statistical theory that satisfies the cluster decomposition principle. This principle states that the measurement of the physical quantities of isolated particles does not depend on the presence of other particles elsewhere far enough. Although it fulfills the cluster decomposition principle, there is no indication yet those fundamental particles present today satisfy parastatistics principles. But this theory remains interesting to investigate further. In particular, many physicists have attempted to obtain related physical quantities for the parastatistics system, in particular, the GCPF [5-9].

Consider a system of particles that do not interact with each other, with the Hamiltonian given by

$$\widehat{H} = \sum_\alpha^m E_\alpha \widehat{N}_\alpha, \tag{3}$$

with E_α is the energy of a single quantum particle state $|i_\alpha\rangle$, $\widehat{N}_\alpha$ is the operator that counts the number of particles in state $|i_\alpha\rangle$, and m is the number of different energy levels (may be degenerate states) denoted by $i_\alpha = 1, \cdots, m$, where m may be infinite.

The GCPF may be written as [10-12]:

$$Z(x_1, \cdots, x_m) = \mathrm{Tr}e^{\beta(\mu\widehat{N} - \widehat{H})}, \tag{4}$$

where $x_i = e^{\beta(\mu - E_i)}$, $\beta = 1/kT$, $T$is the absolute temperature, $k = 1{,}38 \times 10^{-23}\,\mathrm{J/K}$ is the Boltzmann constant, and μ is the chemical potential. The trace covers all states in the system under consideration. GCPF for identical particle system which is invariant to particle permutation has been formulated in [7,13,14], in the form of Schur polynomials sums:

$$Z(x_1, \cdots, x_m) = \sum_{\lambda\in\Lambda} s_\lambda(x_1, \cdots, x_m), \tag{5}$$

where Λ depends upon the type of particle statistics under consideration. Schur's polynomials, $s_\lambda(x_1, \cdots, x_m)$, is a symmetric polynomial which is defined as [15]:

$$s_\lambda(x_1, \cdots, x_m) = \begin{cases} \dfrac{\left|x_j^{m+\lambda_i - i}\right|}{\Delta(x_1, \cdots, x_m)}; \text{for } \lambda_{m+1} = 0, \\ 0 \quad ; \quad \text{else}, \end{cases} \tag{6}$$

with $\Delta(x_1,\cdots,x_m) \equiv \prod(x_i - x_j)$is the Vandermonde polynomials. The form of Schur'spolynomials depend upon the λ parameter, which is a partition from an integer number *n*, that is $\lambda = (\lambda_1 \cdots \lambda_n)$ with $\lambda_i \geq \lambda_{i+1}$, and $\sum_i \lambda_i = n$. Every λ is related to Young's table which consists of boxes in left-justified rows with λ_i box on the *i*-th row. For example, for $n = 11$, and $\lambda = (5,3,1,1,1)$, Young's table is

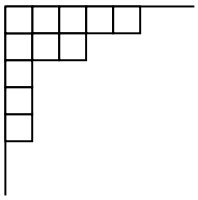

Figure 1. Young's table for $\lambda = (5,3,1,1,1)$.

For paraboson of order *p*, $\bigwedge$ in Eq. (5) consists of sets of all λ having Young's table of a maximum of *p* rows. For parafermi of order *p*, $\bigwedge$ consists of sets of all λ having Young's table of a maximum of *p* column.

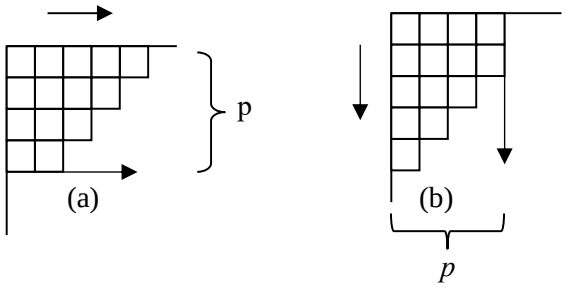

Figure 2.Young's table for $p = 4$ of paraboson (a) and parafermion (b).

GCPF for parafermion of order*p*in Eq. (5), may be simplify into a determinant of two ratios, viz. [5,14]:

$$Z_{(p)}^{pF}(x_1,\cdots,x_m) = \frac{\left|x_j^{m-i} - x_j^{m+p+i-1}\right|}{\left|x_j^{m-i} - x_j^{m+i-1}\right|}. \tag{7}$$

For paraboson of order*p*, the GCPF may also be stated as a determinant of two ratios as [8]:

$$Z_{(p)}^{pB}(x_1,\cdots,x_m) = \frac{\left|P_{(p)}(x_1,\cdots,x_m)\right|}{\left|P_{(0)}(x_1,\cdots,x_m)\right|}. \tag{8}$$

where $\left|P_{(p)}(x_1,\cdots,x_m)\right|$ is the determinant of a matrix with the elements given by

$$\left(P_{(p)}\right)_{ij} = \begin{cases} x_j^{m-i} & ;\text{for } 1 \leq i \leq p, \\ x_j^{m-i} + (-1)^{(p+1)} x_j^{m-p+i-1} & ;\text{untuk}(p+1) \leq i \leq m. \end{cases} \tag{9}$$

$Z_{(p)}^{pB}(x_1,\cdots,x_m)$ in Eq. (8) is obtained via solving a recursion relation which is derived using Schur's polynomial symmetrization formula of [16]:

$$s_\lambda(x_1,\cdots,x_m)=\sum_{i=1}^{m} s_\lambda(x_1,\cdots\hat{x}_i,\cdots,x_m)\prod_{\substack{j=1\\ j\neq i}}^{m}\frac{x_j}{(x_j-x_i)}. \tag{10}$$

Therefore, a recursion relation of parastatistic form is obtained as,

$$\begin{aligned} Z_{(p,q)}^{\mathrm{ind}}(x_1,\cdots,x_m)=\sum_{i=1}^{m} Z_{(p,q)}^{\mathrm{ind}}(x_1,\cdots\hat{x}_i,\cdots,x_m)\prod_{\substack{j=1\\ j\neq i}}^{m}\frac{x_j}{(x_j-x_i)}, \\ +x_1\cdots x_m Z_{(p,q-1)}^{\mathrm{ind}}(x_1,\cdots,x_m) \end{aligned} \tag{11}$$

(with $\hat{x}_i$ means x_i is removed) for any λ partition with $\lambda_m=0$. This equation is obtained from the expansion of minor determinant in Eq. (6). A simple case is exhibited when $p = m$, which previously may be simplified as [17]

$$\boldsymbol{Z_{(p)}^{pB}(x_1,\cdots,x_m)=\frac{1}{\prod_{i=1}^{p}(1-x_i)}\frac{1}{\prod_{\substack{j=1\\ i<j}}^{p}(1-x_ix_j)}}. \tag{12}$$

The above relation may be used as an initial input in the recursion relation of Eq. (11).

An energy level model which is similar to the harmonic oscillator is chosen because this system has energy levels with equal distance, making it easier to calculate. In addition, many physical systems have similar energy levels to harmonic oscillators, i.e.: $E_i=\left(i+\frac{1}{2}\right)\hbar\omega,\ i=0,1,2,\cdots$ such as molecular vibrations, atoms in crystal lattices, and paramagnetic systems. The generalization to different energy levels can be easily made using the input values E_i, provided that the system is a non-interacting particle system.The thermodynamic quantities can be obtained from the complete canonical partition function, *Z*, which is obtained through its complete canonical potential,

$$\Omega(T,V,\mu)=-kTlnZ(T,V,\mu). \tag{13}$$

2. Research Method

This study is conducted in two stages, namely (i) computational method formulation and (ii) numerical casting in a computational programming language. In the first stage of computational method formulation, the study begins with the determination of boundary conditions. Then we analyze the thermodynamic properties of the parastatistics function. The next stage is numerical method casting into the computer program. Before the calculation, the numerical method is tested for its stability against the entered boundary conditions. This is important to be performed so that the result of computer calculation is not a set of data without physical meaning. Additionally, this procedure may also clarify the boundary conditions in the validity of the physical parameters that have been put into the numerics. The average number of particles can be derived as follows:

$$\boldsymbol{N(T,V,\mu)=-\left.\frac{\partial\Omega}{\partial\mu}\right|_{T,V}=\frac{T}{Z}\left.\frac{\partial Z(x_1,\cdots,x_m)}{\partial\mu}\right|_{T}}. \tag{14}$$

The form of $\boldsymbol{\partial Z/\partial\mu}$ is obtained by differentiating the recursion relation of Eq. (11) against $\boldsymbol{\mu}$ at constant temperature *T* and $\boldsymbol{\partial Z/\partial\mu}$ as an initial value in the recursion relation of Eq. (14) obtained from Eq. (11), that is

$$\boldsymbol{\left.\frac{\partial Z(x_1,x_2)}{\partial\mu}\right|_{T}=\beta Z(x_1,x_2)\left[\frac{x_1}{(1-x_1)}+\frac{x_2}{(1-x_2)}+\frac{2x_1x_2}{(1-x_1x_2)}\right]}, \tag{15}$$

where $\boldsymbol{Z(x_1, x_2)}$ is given as in Eq. (12).

3. Results and Discussion

From the computational calculations, we can obtain some graphs that can describe the properties of the thermodynamic functions of the second order paraboson model system. First of all, the area where the allowed values of μ are investigated. This is done by looking at the behavior of *N* which in general must be positive. Therefore, the *N* values arecalculated as a function of μ, which is given in the following graph.

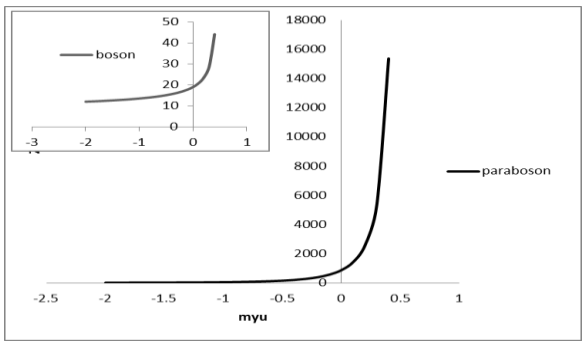

Figure 3. A complete canonical partition function, $Z(\mu)$for paraboson system. Inset: the same quantity for boson.

In Fig. 3 it appears that the complete canonical partition function, Z, for the paraboson system has the same pattern and the value of μ must also be smaller than the lowest energy level of the single particle, $\epsilon = \frac{1}{2}$, that is $\mu < \frac{1}{2}$. Moreover, the value of Z may be investigated, which in general should not be negative as well.

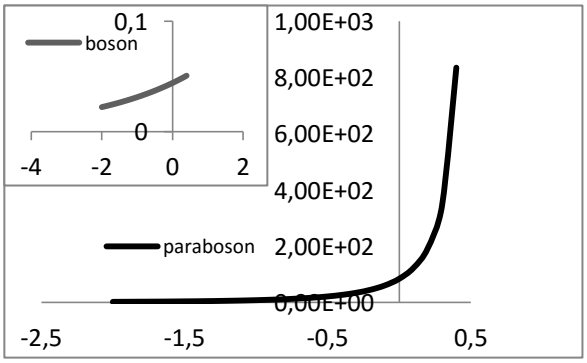

Figure 4. The average number of particles, $N(\mu)$, for paraboson system. Inset: the same quantity for boson.

Fig. 4 above shows that the value of μ is lower than the lowest ϵ. This means that the second-order paraboson system tends to absorb particles like the usual boson system, which tends to increase the number of particles into the system without losing energy (see e.g. Greiner et al. (1995)). From this property alone it can be expected that the second-order paraboson can experience condensation like a boson.

From Fig. 3 for the average number of particles, *N*, for paraboson and parafermion systems, the values of μ must be smaller than the lowest energy level of the single particle, $= \frac{1}{2}$, i.e. $\mu < \frac{1}{2}$. For $\mu \geq \frac{1}{2}$, the value of *N* can become infinitely negative (not shown in the graph).

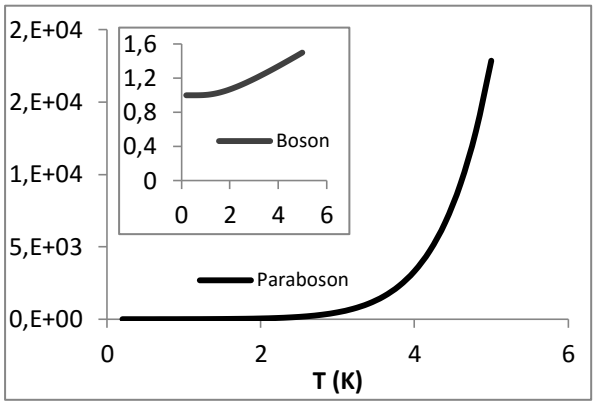

Figure 5. The complete canonical partition function, $Z(T)$, for paraboson system. Inset: the same quantity for boson.

Fig. 5 above shows that the $Z(T)$ pattern for paraboson and boson systems are similar. This means that in both of these systems the complete canonical partition function will increase along with the increase in temperature.

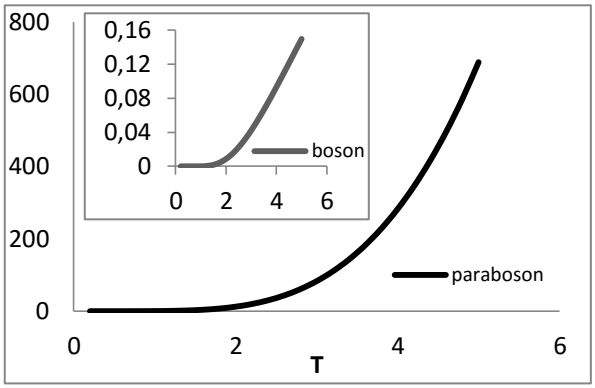

Figure 6. The average number of particles, $N(T)$, for parabosn system. Inset: the same quantity for boson.

Fig. 6 shows that as the temperature, *T*, increases, the value of *N* increases dramatically. This means that with an increase in *T*, the average total number of particles increases as well. The relation between *N* and *T* can be seen in Eq. (14), i.e.: $N \propto T$.

4. Conclusion

The complete canonical partition function, *Z*, for a paraboson system has a pattern similar to boson system. This means that the second-order paraboson system tends to absorb particles like the usual boson system, which tends to increase the number of particles into the system without losing energy. From this property alone it can be expected that the second-order para boson may experience condensation like a boson.

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